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The definite integral of elementary fractional functions and division by zero

According to the Dedekind cut theorem, for a closed interval [0,1], the following relationship between sets holds:

$$\{[0,1]\} = \{[0], (0,1]\} \qquad (1)$$

Note that [0] on the right-hand side of Equation (1) is referred to as a closed point, and is defined by the relationship

$$[0] \equiv [0,0] \qquad (2)$$

This implies that the point is a Euclidean point.

Consider the definite integral of the elementary fractional function

$$-\frac{1}{x^2} \qquad (3)$$

for x over the closed interval [0,1]. In other words, consider

$$\int_0^1 \mathrm{d}x \left(-\frac{1}{x^2}\right) \qquad (4)$$

This definite integral includes division by zero. Therefore, it is not possible to perform definite integration on this function as is. It is necessary to expand this function as shown in the following equation using the relationship in Equation (1). In other words, this becomes

$$\begin{aligned}
\int_{[0}^{1]} \mathrm{d}x \left(-\frac{1}{x^2}\right) &= \int_{[0}^{0]} \mathrm{d}x \left(-\frac{1}{x^2}\right) + \int_{(0}^{1]} \mathrm{d}x \left(-\frac{1}{x^2}\right) \\
&= \left[\frac{1}{x}\right]_{[0}^{0]} + \left[\frac{1}{x}\right]_{(0}^{1]} \\
&= \left(\frac{1}{0} - \frac{1}{0}\right) + \left(1 - \frac{1}{\hat{0}}\right) \\
&= (0 - 0) + (1 - \infty) \\
&= -\infty
\end{aligned}$$

$$\therefore \int_{[0}^{1]} \mathrm{d}x \left(-\frac{1}{x^2}\right) = -\infty \qquad (5)$$

Note that $\hat{0}$ refers to an extremely small positive number that is larger than 0 and smaller than any positive constant. In other words, this can be expressed as

$$0 < \hat{0} \qquad (6)$$

□

Next, consider the definite integral of the hyperbolic function $1/x$ for x over the closed interval [0,1]. In other words,

$$\int_{[0}^{1]} \mathrm{d}x \frac{1}{x} = \int_{[0}^{0]} \mathrm{d}x \frac{1}{x} + \int_{(0}^{1]} \mathrm{d}x \frac{1}{x} \qquad (7)$$

This equation has been expanded using the relationship in Equation (1). Based on this, we obtain

$$\begin{aligned} \int_{[0}^{1]} \mathrm{d}x \frac{1}{x} &= \int_{[0}^{0]} \mathrm{d}x \frac{1}{x} + \int_{(0}^{1]} \mathrm{d}x \frac{1}{x} \\ &= [\log_e x]_{[0}^{0]} + [\log_e x]_{(0}^{1]} \\ &= (\log_e 0 - \log_e 0) + (\log_e 1 - \log_e \hat{0}) \\ &= (\log_e 0 - \log_e 0) + \{0 - (-\infty)\} \\ &= (\log_e 0 - \log_e 0) + \infty \qquad (8) \end{aligned}$$

Incidentally,

$$\int_{[0}^{0]} \mathrm{d}x \frac{1}{x} = \left[\frac{1}{x} \cdot x\right]_{x=0} = \frac{0}{0} = 0 \qquad (9)$$

Conversely,

$$\int_{[0}^{0]} \mathrm{d}x \frac{1}{x} = \log_e 0 - \log_e 0 \qquad (10)$$

Therefore, based on Equation (9) and Equation (10),

$$\log_e 0 - \log_e 0 = 0 \qquad (11)$$

Therefore, based on Equation (8) and Equation (11),

$$\int_{[0}^{1]} \mathrm{d}x \frac{1}{x} = \infty$$

holds. □