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Division by Zero

Proof of 0/0 = 0 (reducible set theory)

Definition: When A and B are real numbers, calculation of division B/A is defined as follows:

In $|B| - (|\{A\}| \cdot |A| + a) = 0(0 \leq a)(A, B, a \in \mathrm{R})$, if $B_0 = B$ and $A_j = A(j = 0, 1, 2, \ldots)$, a set based on A_{j+1} in the reducible recurrence relation, $B_j - A_{j+1} = B_{j+1}$ (this A_{j+1} will be referred to as the j+1$^{\text{th}}$ reducible number) is defined as the j+1$^{\text{th}}$ reducible number set $\{A_{j+1}\}$, and when B_j (this B_j will be referred to as the j^{th} reduced number) satisfies $B_j > B_{j+1} \geq 0$, $\{A_{j+1}\} \neq \emptyset$, and when it does not satisfy this condition, $\{A_{j+1}\} = \emptyset$. The set whose elements belong to the reducible number set $\{A_{j+1}\}$ is defined as a reducible set $\{A\}$.

Here, B is the dividend and A is the divisor. $|\{A\}|$ is the number of elements in the reducible set $\{A\}$, and is the quotient of B/A. a is the remainder, and is the smallest non-negative real number when $|\{A\}|$ is maximized. At this time, the following holds:

Theorem 1: 0/0 = 0.
Proof: Based on the definition $|B| - (|\{A\}| \cdot |A| + a) = 0$, $B = 0 \Rightarrow a = 0$. Therefore,

$$0 - (|\{A\}| \cdot |A| + 0) = 0 - |\{A\}| \cdot |A| = 0$$

where $A = 0 \Rightarrow 0 - |\{A\}| \cdot |A| = 0 - |\{A\}| \times 0 = 0$.

Now, because $B = B_0 = 0 \wedge A = A_j = 0$, the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ is $0 - 0 = 0$, and $B_j > B_{j+1} \geq 0$ is not satisfied. Therefore, the reducible set $\{A\} = \{\{A_1\},\{A_2\},\ldots,\{A_j\},\ldots\} = \{\emptyset,\emptyset,\ldots,\emptyset,\ldots\} = \emptyset$, and the number of elements $|\{A\}| = |\emptyset| = 0$ are obtained. Therefore,

$$A = B = 0 \Rightarrow B - (|\{A\}| \cdot |A| + a) = 0 - (0 \times 0 + 0) = 0$$

$\therefore$ 0/0 = 0.

Proof of 100/0 = 0 (reducible set theory)

Theorem 2: In division B/A, the quotient is 0 when $A = 0$ with the remainder of B.
Proof: When $B > 0$, based on the definition $|B| - (|\{A\}| \cdot |A| + a) = 0$, if $A = 0$, $B = B_0 > 0 \wedge A = A_j = 0$. Therefore, the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ is

$B_0 - 0 = B_0 = B_1$
$B_1 - 0 = B_1 = B_2$
…
$B_j - 0 = B_j = B_{j+1}$

and $B_j > B_{j+1} \geq 0$ is not satisfied.

Therefore, the reducible set $\{A\} = \{\{A_1\},\{A_2\},\ldots,\{A_j\},\ldots\} = \{\emptyset,\emptyset,\ldots,\emptyset,\ldots\} = \emptyset$, and the number of elements $|\{A\}| = |\emptyset| = 0$ is obtained. Therefore,

$$|A| = |\{A\}| \cdot |A| + a = |\emptyset| \cdot |0| + a = 0 \times 0 + a = a$$

holds.

As such, $B>0 \Rightarrow a \neq 0$ is established. Here, $A = 0 \Rightarrow$ quotient$|\{A\}| = 0$, and the remainder $a = B$. Because a is clearly the smallest non-negative number when $|\{A\}|$ is maximized, the quotient division B/A when $A = 0$ and $B \neq 0$ is $|\{A\}|$, with the remainder a being B. Of course, $B = 0 \Rightarrow a = 0$ holds.

Here, the above definition purposely excludes the case in which at least A or B becomes negative; meaning only a quarter plane (the first quadrant) was considered. An extended definition that includes such negative numbers and its effects is shown below:

Extended definition: When A and/or B are real negative number(s), calculation of division B/A is defined as follows:

Quotient $= \frac{|A|}{A}\frac{|B|}{B}|\{A\}|$

Remainder term $= \frac{|B|}{B}a$

If substituted with the above, the result is as follows:

Signs for the quotient in each case are:

i. $A = -\alpha < 0 \wedge B = \beta > 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|-\alpha|}{-\alpha}\frac{|\beta|}{\beta}|\{A\}| = -\frac{\alpha}{\alpha}\frac{\beta}{\beta}|\{A\}| = -|\{A\}|$$

ii. $A = \alpha > 0 \wedge B = -\beta < 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|\alpha|}{\alpha}\frac{|-\beta|}{-\beta}|\{A\}| = -\frac{\alpha}{\alpha}\frac{\beta}{\beta}|\{A\}| = -|\{A\}|$$

iii. $A = -\alpha < 0 \wedge B = -\beta < 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|-\alpha|}{-\alpha}\frac{|-\beta|}{-\beta}|\{A\}| = -\frac{\alpha}{\alpha}\frac{\beta}{\beta}|\{A\}| = -|\{A\}|$$

iv. $A = 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|0|}{0}\frac{|\beta|}{\beta}|\{A\}| = 0 \times 1 \times |\{A\}| = 0 \times |\{A\}|$$

v. $B = 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|\alpha|}{\alpha}\frac{|0|}{0}|\{A\}| = 1 \times 0 \times |\{A\}| = 0 \times |\{A\}|$$

vi. $A = 0 \wedge B = 0$,

$$\text{Quotient} = \frac{|A|}{A}\frac{|B|}{B}|\{A\}| = \frac{|0|}{0}\frac{|0|}{0}|\{A\}| = 0 \times 0 \times |\{A\}| = 0 \times |\{A\}|$$

Signs for the remainder term in each case are:

I. $A = \alpha > 0 \wedge B = \beta > 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|\beta|}{\beta}\alpha = \frac{\beta}{\beta}\alpha = \alpha$$

II. $A = -\alpha < 0 \wedge B = \beta > 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|\beta|}{\beta}\alpha = \frac{\beta}{\beta}\alpha = \alpha$$

III. $A = \alpha > 0 \wedge B = -\beta < 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|-\beta|}{-\beta}\alpha = -\frac{\beta}{\beta}\alpha = -\alpha$$

IV. $A = -\alpha < 0 \wedge B = -\beta < 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|-\beta|}{-\beta}\alpha = -\frac{\beta}{\beta}\alpha = -\alpha$$

V. $A = 0 \wedge B = \pm\beta \neq 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|\pm\beta|}{\pm\beta}\alpha = \pm 1 \times \alpha = \pm\alpha$$

VI. $A \neq 0 \wedge B = 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|0|}{0}\alpha = 0 \times \alpha = 0$$

VII. $A = 0 \wedge B = 0$

$$\text{Remainder term} = \frac{|B|}{B}\alpha = \frac{|0|}{0}\alpha = 0 \times \alpha = 0$$

Lemma 1: In division B/A, if $A > B = 0$, based on the definition, $B = B_0 = 0 \wedge A = A_j > 0$, and the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ is

$B_0 - A_1 = 0 - A = B_1$
$B_1 - A_2 = -A - A = -2A = B_2$
…
$B_j - A_{j+1} = -(j+1)A = B_{j+1}$

and $B_j > B_{j+1} \geq 0$ is not satisfied.

Therefore, the reducible set $\{A\} = \{\{A_1\},\{A_2\},\ldots,\{A_j\},\ldots\} = \{\emptyset,\emptyset,\ldots,\emptyset,\ldots\} = \emptyset$, and the number of elements $|\{A\}| = |\emptyset| = 0$ are obtained. Therefore, when $A > B = 0$,

$$0 - (|\{A\}| \cdot |A| + a) = 0 - (|\emptyset| \cdot |A| + a) = 0 - (0 \times |A| + a) = 0 - (0 + a) = 0$$

$\therefore a = 0$.

As such, $A > B = 0 \Rightarrow$ quotient $|\{A\}| = 0$ and remainder $a = 0$ are established.

Lemma 2: In division B/A, if $A > B > 0$, based on the definition, $0 < B = B_0 < A = A_j$, and the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ is

$$B_0 - A_1 = 0 - A = B_1$$
$$B_1 - A_2 = -A - A = -2A = B_2$$
$$\ldots$$
$$B_j - A_{j+1} = -jA - A = -(j+1)A = B_{j+1}$$

and $B_j > B_{j+1} \geq 0$ is not satisfied.

Therefore, the reducible set $\{A\} = \{\{A_1\},\{A_2\},\ldots,\{A_j\},\ldots\} = \{\emptyset,\emptyset,\ldots,\emptyset,\ldots\} = \emptyset$, and the number of elements $|\{A\}| = |\emptyset| = 0$ are obtained.

Therefore, when $A > B > 0$,

$$B - (|\{A\}| \cdot |A| + a) = B - (|\emptyset| \cdot |A| + a) = B - (0 \times |A| + a) = B - (0 + a) = B - a = 0$$

$\therefore B = a$ is obtained.

As such, $A > B > 0 \Rightarrow$ quotient $|\{A\}| = 0$ and remainder $a = B$ are established.

Lemma 3: In division B/A, if $B > A = 0$, $A = 0 \Rightarrow$ quotient $|\{A\}| = 0$ and remainder $a = B$ are established.
It is proven by Theorem 2.

Lemma 4: In division B/A, if $B > A > 0$, based on the definition, dividend B can be expressed as $B = kA + b$ (however, $k \in \mathrm{N}$, $0 \leq b < A$).

Using this expression, we obtain $B_0 = B = kA + b$; thus, the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ is:

$$B_0 - A_1 = (kA + b) - A = \{(k-1)A + b\} = B_1$$
$$B_1 - A_2 = \{(k-1)A + b\} - A = \{(k-2)A + b\} = B_2$$
$$\ldots$$
$$B_{k-1} - A_k = (A + b) - A = b = B_k$$
$$B_k - A_{k+1} = b - A = B_{k+1} < 0$$

and $B_0 > B_1 > \cdots > B_{k-1} > B_k \geq 0$ is clearly established from the hypothesis. $B_k > B_{k+1} \geq 0$ is not satisfied.

Therefore, the reducible number set $\{A_{j+1}\}$ is

$$\{A_1\} \neq \emptyset \wedge \{A_2\} \neq \emptyset \wedge \cdots \wedge \{A_k\} \neq \emptyset \wedge \{A_{k+1}\} = \emptyset \wedge \{A_{k+2}\} = \emptyset \wedge \cdots$$

and the reducible set $\{A\}$ is

$$\{A\} = \{\{A_1\} \neq \emptyset, \{A_2\} \neq \emptyset, \cdots, \{A_k\} \neq \emptyset, \{A_{k+1}\} = \emptyset, \{A_{k+2}\} = \emptyset, \cdots\}$$

The number of elements $|\{A\}|$ of the reducible set $\{A\}$ clearly becomes $|\{A\}| = k$. In other words, the quotient of the division B/A is k, and this is consistent with the hypothesis.

If this result is applied to the definition:

$$|B| - (|\{A\}| \cdot |A| + a) = B - (kA + a) = (kA + b) - (kA + a) = b - a = 0$$

$\therefore a = b$ $(0 \leq b < A)$ is established.

Here, if $b = 0$, B is clearly a positive real number and is an integral multiple of A. This means that if the remainder a is 0 and $b > 0$, and if B is divided by A, the remainder a becomes b.

Lemma 5: In division B/A, if $A = B = 0$, $0/0 = 0$.
It is proven by Theorem 1.