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Generalized Euler's Principle and Division by Zero

Definition The extended definition of Euler's principle is as follows. Starting from

$$e_r^{i\theta} = \cos_r \theta + i \sin_r \theta$$

each component of the right-hand side can be defined as

$$\cos_r \theta \equiv \frac{x_r}{r}, \quad \sin_r \theta \equiv \frac{y_r}{r}$$

Thus, when $r \neq 0$,

$$\cos_{r\neq 0} \theta \equiv \cos\theta, \quad \sin_{r\neq 0} \theta \equiv \sin\theta$$

and $r = 0 \Rightarrow \theta = 0$, in which case

$$\cos_0 \theta = \frac{x_0}{0} = \frac{0}{0} = 0, \quad \sin_0 \theta = \frac{y_0}{0} = \frac{0}{0} = 0$$

However, this extended definition depends on the fundamental theorem of division by zero, 0/0 = 0. Based on the above definitions, we have the following theorem.

Theorem The following holds true:

$$e_r^0 = \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases}$$

Proof Based on the definition

$$e_r^{i\theta} = \cos_r \theta + i \sin_r \theta \qquad (1)$$

when $r \neq 0$, letting $\theta = 0$ yields

$$\cos_{r\neq 0} \theta = \cos_{r\neq 0} 0 = \cos 0 = 1 \qquad (2)$$

and

$$\sin_{r\neq 0} \theta = \sin_{r\neq 0} 0 = \sin 0 = 0 \qquad (3)$$

Substituting (2) and (3) into (1) yields

$$e_r^{i\theta} = e_{r\neq 0}^0 = \cos_{r\neq 0} \theta + i \sin_{r\neq 0} \theta = \cos 0 + i \sin 0 = 1 + 0 = 1 \qquad (4)$$

When $r = 0$, then $r = 0 \Rightarrow \theta = 0$, in which case

$$\cos_r \theta = \cos_0 0 = \frac{x_0}{0} = \frac{0}{0} = 0 \qquad (5)$$

and

$$\sin_r \theta = \sin_0 0 = \frac{y_0}{0} = \frac{0}{0} = 0 \qquad (6)$$

Substituting (5) and (6) into (1) yields

$$e_r^{i\theta} = e_{r=0}^{0} = \cos_{r=0} \theta + i \sin_{r=0} \theta = \cos_0 0 + i \sin_0 0 = 0 + 0 = 0 \qquad (7)$$

Based on the above, combining (4) and (7) yields

$$e_r^0 = \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases} \qquad (8)$$

QED.

In addition, by definition, we have

$$e_{r\neq 0}^{i\theta} = \cos_{r\neq 0} \theta + i \sin_{r\neq 0} \theta = \cos\theta + i \sin\theta = e^{i\theta} \qquad (9)$$