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Hiroshi Michiwaki

General long division method and division by zero

1. Introduction

Division is a calculation between two numbers a and b in which we ask how many times b can be subtracted from a. Relatively rational calculation of such an operation generally uses multiplication, which is considered the inverse operation of division. However, this is not the real solution for division; it is simply a general method.

For example, when calculating $a \div b = 15 \div 3$,

i. First, let us think about $3 \times 4 = 12$. However, this leaves a remainder. Thus,
ii. Next, as a value slightly larger than 4, we select 6, and calculate $3 \times 6 = 18$. This time, it was a little too large, so
iii. Finally, we find that $3 \times 5 = 15$. The product of the remainder and the arbitrary selected number were consistent with the dividend, thus, we select 5 as the solution of $15 \div 3$.

This example is quite simple, but sufficiently shows the nature of division. In other words, essentially, we conduct subtraction operations as follows:

$15 - 3 = 12$ (first time)
$12 - 3 = 9$ (second time)
$9 - 3 = 6$ (third time)
$6 - 3 = 3$ (fourth time)
and
$3 - 3 = 0$ (fifth time)

However, this operation is cumbersome. Thus, up to an arbitrary number of operations that is the likely solution, we skip ahead using multiplication for rationalization. In other words, the above operations i to iii mean:

i. $15 - 3 \times 4 = 3$
ii. $15 - 3 \times 6 = -3$
iii. $15 - 3 \times 5 = 0$

This means that multiplication, which has long been used as a historical solution method for division, is an operation in which division, defined as repeated subtraction, is abridged by the use of the product of divisor and solution, or divisor and quasi-solution, such as 4 or 6 in the above example, for part of the whole calculation, without using repeated subtraction.

As the example shows, defining division as an inverse operation of multiplication causes the confusion that it is the only definition; however, in truth, it is more natural to think that solving division by multiplication is an extension of a shortcut involving repeated subtractions. Division was not originally defined as an inverse operation of multiplication, and it should be defined independently of multiplication. Its independent definition is “to find a solution that is the number of times a divisor can be subtracted from a dividend, to obtain

the smallest non–negative number as a resulting value.” This is the original definition of division.

Based on this true definition of division, we show the manual method (general long division) of division below.

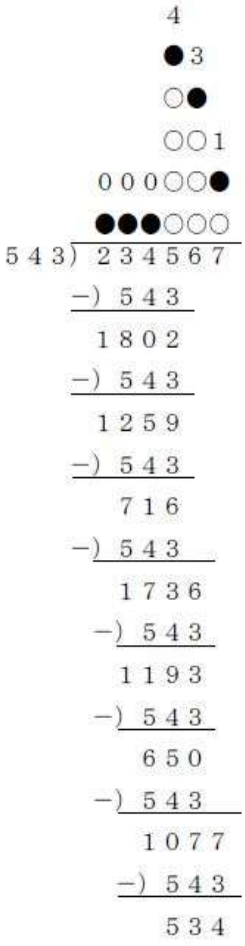
2. General long division 1

Definition
In the manual method of division of a dividend consisting of an arbitrary number of digits, for a certain digit of the dividend, the divisor is subtracted once from the said dividend, and as a result, the dividend is reduced. If the result of reduction is a non-negative value, then above the applicable dividend digit, stack ○ upward. Continue this operation as long as possible, adding ○s in the rows above.

If subtracting the divisor from the dividend makes no difference, or if it leads to a negative number, add ●, then move to the next smaller digit, and repeat the above steps.

When this repeat operation is completed, count the number of ○s for each digit, which is the quotient for the applicable digit. ● is 0.

Example 1: General long division I for 234567÷543 = 431...534 is shown below.



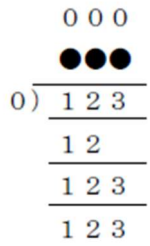
1) If divisor 543 is subtracted from the target dividend 2, the result is negative; therefore, ● is placed immediately above the target dividend 2, and 0 is placed above (vacant position, 0), then the next digit is considered.
2) If divisor 543 is subtracted from the target dividend, which is now 23, the result is negative; therefore, ● is placed immediately above the target dividend 3, and 0 above (vacant position, 0), then the next digit is considered.
3) If divisor 543 is subtracted from the target dividend, now 234, the result is negative; therefore, ● is placed immediately above the target dividend 4, and 0 above (vacant position, 0), then the next digit is considered.
4) If divisor 543 is subtracted from the target dividend, now 2345, the result is non-negative; therefore, ○ is placed immediately above the target dividend 5. From the target dividend 2345, divisor 543 is subtracted once, and as in the usual manual subtraction, a line is drawn under the applicable divisor, and the subtracted result of 1802 is placed below the line.
5) If divisor 543 is subtracted from 1802, the result is non-negative; therefore, ○ is placed immediately above ○ above the target dividend 5 in a stack, and 543 is subtracted from 1802 once. Below this, the subtracted result of 1259 is placed.
6) If divisor 543 is subtracted from 1259, the result is non-negative; therefore, ○ is placed immediately above ○ above the target dividend 5 in the stack, and 543 is subtracted from 1259 once. Below this, the subtracted result of 716 is placed.
7) If divisor 543 is subtracted from 716, the result is non-negative; therefore, ○ is placed immediately above the top ○ above the target dividend 5 in the stack, and 543 is subtracted from 716. Below this, the subtracted result of 173 is placed.
8) If divisor 542 is subtracted from 173, the result is negative; therefore, above the top ○, ● is placed in a stack. Above this, number 4 is placed, which is the number of ○s, then the next digit is considered.
9) If divisor 543 is subtracted from the target dividend, now 1736, the result is non-negative; therefore, ○ is placed immediately above the target dividend 6, and 543 is subtracted from 1736 once, and as in the usual manual subtraction, a line is drawn under the applicable divisor, and the subtracted result of 1193 is placed below the line.
10) If divisor 543 is subtracted from 1193, the result is non-negative; therefore, ○ is placed immediately above the top ○ of the target dividend 6 in a stack, and divisor 543 is subtracted from 1193. Below this, the subtracted result of 650 is placed.
11) If divisor 543 is subtracted from 650, the result is non-negative; therefore ○ is placed immediately above the top ○, and the divisor 543 is subtracted from 650 once. Below this, the subtracted result of 107 is placed.
12) If divisor 543 is subtracted from 107, the result is negative; therefore, ● is placed above the top ○ immediately above the target dividend of 6. Above this, a 3 is placed, which is the number of ○s, then the next digit is considered.
13) If divisor 543 is subtracted from the target dividend, now 1077, the result is non-negative; therefore, ○ is placed immediately above the target dividend of 7, and the divisor 543 is subtracted from 1077 once, and as in the usual manual subtraction, a

line is drawn under the applicable divisor, and the subtracted result of 534 is placed below the line.

14) If divisor 543 is subtracted from 534, the result is negative; therefore, ● is placed immediately above the top ○ above the target dividend of 7. Above this, the number of ○s is placed, which is 1, then the next digit is considered. Because the operation within the integer range is completed, the calculation is stopped here with 534 as the remainder. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 7) and calculation below the decimal point can be continued.

15) In other words, the solution of this division is 000431, with a remainder of 534; however, according to mathematical custom, 0 above the largest digit can be omitted: thus, the quotient is 431 with a remainder of 534.

Example 2: General long division I for 123÷0 = 0...123 is shown below.



1. If divisor 0 is subtracted from the target dividend 1, it remains at 1 without change; therefore, ● is placed immediately above the target dividend value of 1, and 0 is placed above (vacant space, 0). A line is drawn below the target dividend of 1, and 1 is placed below the line, and the next digit is considered.
2. If divisor 0 is subtracted from the target dividend, now 12, it remains at 12 without a change; therefore, ● is placed immediately above the target dividend of 2, and 0 is placed above (vacant space, 0). A line is drawn below the target dividend of 12, and 12 is placed below the line, and the next digit is considered.
3. If divisor 0 is subtracted from the target dividend, now 123, it remains at 123 without a change; therefore, ● is placed immediately above the target dividend of 3, and 0 is placed above (vacant space, 0). Then moving to the next digit, a line is drawn below the target dividend of 123, and 123 is placed under the line. At this point, because the operation within the integer range is completed, calculation is stopped with the remainder of 123. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 3), and calculation can be continued to an arbitrary digit below the decimal point.

3. General long division II

Definition
○ and ● in general long division I are replaced with 1 and 0, respectively, and as quotient value for each digit, above the first 0, the sum of the numbers in the rows below (0 to 9) is placed.

Example 3: General long division II of 234567÷543=431...534 is shown below.

```
          4
          0 3
          1 0
          1 1 1
          1 1 0
        0 0 0 1 1 1
5 4 3) 2 3 4 5 6 7
     −) 5 4 3
        1 8 0 2
     −) 5 4 3
        1 2 5 9
     −) 5 4 3
          7 1 6
     −) 5 4 3
          1 7 3 6
       −) 5 4 3
          1 1 9 3
       −) 5 4 3
            6 5 0
       −) 5 4 3
          1 0 7 7
        −) 5 4 3
            5 3 4
```

1. If divisor 543 is subtracted from the target dividend 2, the result is negative; therefore, 0 is placed immediately above the target dividend of 2 (vacant space, 0), and the next digit is considered.
2. If divisor 543 is subtracted from the target dividend, now 23, the result is negative; therefore, 0 is placed immediately above the target dividend of 3 (vacant space, 0), and the next digit is considered.
3. If divisor 543 is subtracted from the target dividend, now 234, the result is negative; therefore, 0 is placed above the target dividend of 4 (vacant space, 0), and the next digit is considered.
4. If divisor 543 is subtracted from the target dividend, now 2345, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 5. Divisor 543 is subtracted from 2345 once, and as in the usual manual subtraction, a line is drawn below the target dividend, and the subtraction result 1802 is placed below the line.
5. If divisor 543 is subtracted from 1802, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 5 in a stack. Divisor 543 is subtracted from 1802 once, and the subtraction result of 1259 is placed below the line.
6. If divisor 543 is subtracted from 1259, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 5 in a stack. Divisor 543 is subtracted from 1259 once, and the subtraction result of 716 is placed below the line.
7. If divisor 543 is subtracted from 716, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 5 in a stack. Divisor 543 is subtracted from 716 once, and the subtraction result of 173 is placed below the line.
8. If divisor 543 is subtracted from 173, the result is negative; therefore, 0 is placed immediately above the target dividend of 5 and above the top 1 in a stack. Above this 0, a 4 is placed, which is the sum of the rows below 0. Then, the next digit is considered.
9. If divisor 543 is subtracted from the target dividend, now 1736, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 6. Divisor

543 is subtracted from 1736 once. As in the usual manual subtraction, a line is drawn below the target dividend, and the subtraction result of 1193 is placed below the line.

10. If divisor 543 is subtracted from 1193, the result is non-negative; therefore, 1 is placed immediately above the target dividend of 6 in a stack. Divisor 543 is subtracted from 1193 once, and the subtraction result of 650 is placed below. If divisor 543 is subtracted from 650, the result is non-negative; therefore, 1 is placed immediately above the target dividend 6 above the top 1 in a stack. Divisor 543 is subtracted from 650 once, and the subtraction result of 107 is placed below.
11. If divisor 543 is subtracted from the target dividend, now 107, the result is negative; therefore, 0 is placed immediately above the target dividend of 6 above the top 1, and above the 0, a 3 is placed, which is the sum of the rows below 0. Then the next digit is considered.
12. If divisor 543 is subtracted from the target dividend, now 1077, the result is non-negative; therefore 1 is placed immediately above the target dividend of 7. Divisor 543 is subtracted from 1077 once, and as in the usual manual subtraction, a line is drawn below the target dividend, and the subtraction result of 534 is placed below.
13. If divisor 543 is subtracted from 534, the result is negative; therefore, 0 is placed immediately above the target dividend of 7 above the top 1. Above 0, place 1, which is the sum of the rows below 0. Then we move to the next digit. Because operation in the integer range is complete at this point, calculation is stopped with the remainder of 534. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 7) and calculation can be continued to an arbitrary digit below the decimal point.
14. The quotient is 000431 and the remainder is 534, but the 0 above the largest digit is omitted: the quotient is 431 and the remainder is 534.

Example 4: General long division I of division 123÷0 = 0...123 is shown below.

```
   000
0)123
  12
  123
  123
```

1)If divisor 0 is subtracted from the target dividend of 1, the result remains 1 without a change; therefore, 0 is placed immediately above the target dividend of 1 (vacant space, 0). A line is drawn under the target dividend of 1, and the subtraction result of 1 is placed below the line. Then the next digit is considered.

2)If divisor 0 is subtracted from the target dividend, now 12, the result remains 12 without a change; therefore, 0 is placed immediately above the target dividend of 2 (vacant space, 0). A line is drawn under the target dividend of 12, and the subtraction result of 12 is placed below the line. Then the next digit is considered.

3) If divisor 0 is subtracted from the target dividend, now 123, the result remains 123 without a change; therefore, 0 is placed immediately above the target dividend of 3. A line is drawn below the target dividend of 123, and the subtraction result of 123 is placed below the line. At this point, the operation within the integer range is complete; thus, calculation is stopped with the remainder of 123. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 3) and calculation can be continued to an arbitrary digit below the decimal point.
4) The solution of this division is a quotient of 000 with a remainder of 123. However, according to the mathematical custom, 0s before the largest digit are omitted: a quotient of 0 with a remainder of 123.

4. General long division III

Here, as general long division, we present a manual method that partially expands general long division II compared to the traditional manual method of division. As we discussed in the introduction, in usual division, subtractions are repeated as an essential operation of division; however, to omit this operation somewhat rationally, multiplication is used to skip ahead to the likely solution to compress the subtraction operation. This is the same in the manual method of division.

As an example, let us follow the division 234567÷543 discussed in general long division I and II in steps.

Example 5:

```
        4 3 1
5 4 3) 2 3 4 5 6 7
   −) 2 1 7 2
       1 7 3 6
    −) 1 6 2 9
       1 0 7 7
     −)   5 4 3
         5 3 4
```

1) Considering the target dividend 2 and divisor 543, 2 < 543; thus we move to the next digit.
2) Considering the target dividend 23 and divisor 543, 23 < 543; thus we move to the next digit.
3) Considering the target dividend 234 and divisor 543, 234 < 543; thus we move to the next digit.
4) Considering the target dividend 2345 and divisor 543, 2345 ≥ 543; thus, we select a quasi-solution of 4 where the product of multiplying the value of a suitable digit (quasi-solution number) and divisor 543 would likely be equal to or slightly less than the target dividend of 2345, and place this quasi-solution immediately above the target multiplicand of 5. After placing the product 2172 immediately below the target dividend 2345, a line is drawn under the product 2172, and the subtraction result of 173 is placed below the line. Then we move to the next digit.
5) Considering the target dividend of 1736 and divisor 543, 1736 ≥ 543; thus, we select a quasi-solution of 3 where the product of the arbitrary quasi-solution and divisor 543 would likely be equal to or slightly less than the target dividend of 1736, and place this quasi-solution immediately above the target multiplicand of 6. After placing the product 1629 immediately below the target dividend 1736, a line is drawn under the product 1629, and the subtraction result of 107 is placed below the line. Then we move to the next digit.
6) Considering the target dividend 1077 and divisor 543, 1077 ≥ 543; thus, we select a quasi-solution of 1 where the product of the arbitrary quasi-solution and divisor 543 would likely be equal to or slightly less than the target dividend of 1077, and place this quasi-solution immediately above the target multiplicand of 7. After placing the product 543 immediately below the target dividend 1077, a line is drawn under the product 543, and the subtraction result of 534 is placed below the line. At this point, operation within the integer range is completed; thus, calculation is stopped with the subtraction result of 534 as the remainder. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 7) and calculation can be continued to an arbitrary digit below the decimal point. The quotient is 431 and the remainder is 534. The quotient of 431 should show 000431 to indicate vacant spaces as we moved down the digits, but these 0s are omitted.

Now, as we can see from the operation procedure of the usual manual method of division (especially 4), 5), and 6)), selecting the quasi-solution is simply selecting the optimum

number by multiplying the divisor with several arbitrary one-digit numbers as a trial calculation. This is because we wouldn't know how many times a divisor can be subtracted in the repeated operation until the operation is performed. In other words, deciding the solution for each digit is just selecting an appropriate number that works in multiplication as an inverse operation.

Thus, let us show a calculation method that is able to move the calculation operation in long division in a rational manner by introducing procedures of general long division II into the usual manual method of division.

Example 6:
Step 1.
First, instead of forcefully selecting a quasi-solution, choose a quasi-solution with a sufficiently large remainder, and place a quasi-solution of 3 immediately above the target dividend of 5. Place the product of the divisor 543 and the quasi-solution of 3, 1629, immediately below the target dividend of 2345. A line is drawn under this product, and the subtraction result of 716 is placed under the line.

```
        3
543)234567
  -) 1629
     716
```

Step 2.
Next, because the subtraction result above is 716, and 716 ≥ 543, subtracting the divisor 543 from 716 results in non-negative number. Therefore, 1 is placed immediately above the quasi-solution of 3 in a stack. The divisor 543 is subtracted from 716 once, and the subtraction result of 173 is placed below. Then we move to the next digit.

```
        1
        3
543)234567
  -) 1629
     716
   -) 543
     1736
```

Step 3.
Next, though the target dividend of 1736 means 1736 ≥ 543, we would not force the selection of a quasi-solution. Instead, we choose a quasi-selection with a sufficiently large remainder. We place a quasi-solution of 2 immediately above the target dividend of 6. The product of the divisor 543 and the quasi-solution 2, 1086, is placed immediately below the target dividend 1736. A line is drawn below, and the subtraction result of 650 is placed under the line.

```
          1
          3 2
5 4 3) 2 3 4 5 6 7
   −) 1 6 2 9
        7 1 6
     −) 5 4 3
          1 7 3 6
       −) 1 0 8 6
            6 5 0
```

Step 4.
Next, because the subtraction result above is 650 and 650 ≥ 543, if we subtract the divisor of 543 from the subtraction result of 650, the result is non-negative. Therefore, we place 1 immediately above the above quasi-solution of 2 in a stack. We subtract the divisor of 543 from the subtraction result of 650 once, and place the subtraction result of 107 below. Then we move to the next digit.

```
          1 1
          3 2
5 4 3) 2 3 4 5 6 7
   −) 1 6 2 9
        7 1 6
     −) 5 4 3
          1 7 3 6
       −) 1 0 8 6
            6 5 0
         −) 5 4 3
            1 0 7 7
```

Step 5.
Next, though the target dividend is 1077, and 1077 ≥ 543, we do not forcefully select a quasi-solution, but find a quasi-solution with sufficiently large remainder. We place the quasi-solution of 1 immediately above the target dividend of 7. The product of the divisor 543 and the quasi-solution of 1, 543, is placed immediately below the target dividend of 1077. A line is drawn below, and the subtraction result of 534 is placed below the line.

```
          1 1
          3 2 1
5 4 3) 2 3 4 5 6 7
   −) 1 6 2 9
        7 1 6
     −) 5 4 3
        1 7 3 6
     −) 1 0 8 6
          6 5 0
      −) 5 4 3
          1 0 7 7
       −) 5 4 3
          5 3 4
```

Step 6.
Next, because the target dividend is 534, and 543 < 543, we move to the next digit. Here, operation within the integer range is completed, thus the calculation is stopped with the subtraction result of 534 as the remainder. Of course, a decimal point can be added to the quotient (to the right of the target dividend digit 7) and calculation can be continued to an arbitrary digit below the decimal point. We draw a line above the top row of the quasi-solution number and present the sum of the quasi-solution numbers above this line.

```
          4 3 1
          1 1
          3 2 1
5 4 3) 2 3 4 5 6 7
   −) 1 6 2 9
        7 1 6
     −) 5 4 3
        1 7 3 6
     −) 1 0 8 6
          6 5 0
      −) 5 4 3
          1 0 7 7
       −) 5 4 3
          5 3 4
```

As shown above, the solution of this division is a quotient of 431 with a remainder of 534.

If we follow the above procedure, the manual method of division can rationally progress in accordance with the traditional method. Traditionally, it was common to erase the solution

when one selected an incorrect (not optimal) quasi-solution, but with this method, such a process is unnecessary. If a quasi-solution is not too small but too large, by handling the quasi-solution number added above other quasi-solutions as a negative number instead of a positive number, these numbers can be cancelled. However, in such a case, instead of subtracting the number below from the number above, these numbers are added, which makes the calculation confusing. Therefore, the initial quasi-solution should be relatively small.